

A Hecke algebra attached to mod 2 modular forms of level 5

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Abstract

Let F be the element $\sum_{n \text{ odd}, n>0} x^{n^2}$ of $Z/2[[x]]$. Set $G = F(x^5)$, $D = F(x) + F(x^{25})$. For $k > 0$, $(k, 10) = 1$, define D_k as follows. $D_1 = D$, $D_3 = D^8/G$, $D_7 = D^2G$, $D_9 = D^4G$; furthermore $D_{k+10} = G^2D_k$.

Using modular forms of level $\Gamma_0(5)$ we show that the space W spanned by the D_k is stabilized by the formal Hecke operators $T_p : Z/2[[x]] \rightarrow Z/2[[x]]$, $p \neq 2$ or 5 . And we determine the structure of the (completed) shallow Hecke algebra attached to W . This algebra proves to be a power series ring in T_3 and T_7 with an element of square 0 adjoined. As Hecke module, W identifies with a certain subquotient of the space of mod 2 modular forms of level $\Gamma_0(5)$, and our Hecke algebra result parallels findings in level 1 (by J.-L. Nicolas and J.-P. Serre) and in level $\Gamma_0(3)$ by us.

1 Some spaces of mod 2 modular forms of level $\Gamma_0(5)$

Nicolas and Serre [3], [4] have proved various results about the action of the Hecke algebra on the space of mod 2 modular forms of level 1. In [1] we gave a variant of their results in level $\Gamma_0(3)$. Here we find close analogues in level $\Gamma_0(5)$.

We first summarize results from [3], [4] just as we did at the start of [1]. There are commuting formal Hecke operations $T_p : Z/2[[x]] \rightarrow Z/2[[x]]$, one for each odd prime p . Here $T_p(\sum c_n x^n) = \sum c_{pn} x^n + \sum c_n x^{pn}$. Let F in $Z/2[[x]]$ be $\sum_{n \text{ odd}, n>0} x^{n^2}$. Using modular forms of level 1, Nicolas and Serre show that the T_p stabilize the space spanned by F, F^3, F^5, F^7, \dots and that the associated (completed) Hecke algebra is a power series ring in T_3 and T_5 . Indeed they make the space into a faithful $Z/2[[X, Y]]$ -module with X and Y acting by T_3 and T_5 , and show that each T_p is multiplication by an element of the maximal ideal (X, Y) .

In [1] we took D in $Z/2[[x]]$ to be $\sum_{(n,6)=1, n>0} x^{n^2}$, so that $D = F(x) + F(x^9)$. $W1$ had a basis consisting of the D^k with $k \equiv 1 \pmod{6}$ and $W5$ a basis consisting of the D^k , $k \equiv 5 \pmod{6}$; W was $W1 \oplus W5$. We showed that the T_p with $p \equiv 1 \pmod{6}$ stabilize $W1$ and $W5$, while when $p \equiv 5 \pmod{6}$, $T_p(W1) \subset W5$ and $T_p(W5) \subset W1$. We further showed that $W1$ has a basis $m_{i,j}$ “adapted to T_7 and T_{13} ” with $m_{0,0} = D$. We deduced that the (completed shallow) Hecke algebra attached to W is a power series ring in T_7 and T_{13} with an element of square 0 adjoined. Though W is a space of mod 2 modular forms of level $\Gamma_0(9)$, it identifies as Hecke-module with a certain subquotient of the space of odd mod 2 modular forms of level $\Gamma_0(3)$.

In the present work we change notation. Now $D = \sum_{(n,10)=1, n>0} x^{n^2}$, so that $D = F(x) + F(x^{25})$. Let $G = F(x^5)$. For $k > 0$ with $(k, 10) = 1$ define D_k as follows: $D_1 = D$, $D_3 = D^8/G$, $D_7 = D^2G$ and $D_9 = D^4G$, while $D_{k+10} = G^2D_k$. Let W be spanned by the D_k . (As $D_k = x^k + \dots$, these are linearly independent over $Z/2$.) Then $W = W_a \oplus W_b$ where the D_k , $k \equiv 1, 3, 7, 9 \pmod{20}$, are a basis of W_a , and the D_k , $k \equiv 11, 13, 17, 19 \pmod{20}$ are a basis of W_b . We will establish the following analogues to the results of [1].

- (1) The T_p , $p \neq 5$, stabilize W . If $p \equiv 1, 3, 7, 9 \pmod{20}$, T_p stabilizes W_a and W_b , while if $p \equiv 11, 13, 17, 19 \pmod{20}$, $T_p(W_a) \subset W_b$ and $T_p(W_b) \subset W_a$.
- (2) Though W is a space of mod 2 modular forms of level $\Gamma_0(25)$, it identifies as Hecke-module with a certain subquotient of the space of odd mod 2 modular forms of level $\Gamma_0(5)$.
- (3) W_a admits a basis $m_{i,j}$ adapted to T_3 and T_7 in the sense of Nicolas and Serre, with $m_{0,0} = D$.
- (4) The (completed shallow) Hecke algebra attached to W is a power series ring in T_3 and T_7 with an element of square 0 adjoined.

The proofs of (1) and (2) occupy sections 1 and 2. Some ingredients are the mod 2 level 5 modular equation $(F + G)^6 = FG$ of Theorem 1.12 and the relation $D^{15} + G^4D^3 + G^3 = 0$ of Lemma 2.4.

The proofs of (3) and (4) resemble those of corresponding results in [1]. We identify W_a with a subspace V' of the polynomial ring $Z/2[w]$, making D_k correspond to w^k , and show that under this identification $T_3(D_k)$ corresponds to a certain P_k defined in [2], with $P_{k+80} = w^{80}P_k + w^{20}P_{k+20}$. Lemma 5.5 of [2], which deals with this recursion, gives insight into the action of T_3 . Suppose in particular that q is a power of 2, and let $W_a(q)$ be spanned by the D_k in W_a with $k < 40q^2$. Using this Lemma 5.5 we find that the kernel of $T_3 : W_a(q) \rightarrow W_a(q)$ has dimension at most $2q$. Ideal theory in $Z[\sqrt{-10}]$, developed in section 4, then shows that this kernel is a space $DI(q)$ of theta-series attached to binary quadratic forms. A study of the action of T_7 on $DI(q)$, together with formalism from [1], leads, in section 5, to a proof of (3). And the further study in section 6 of $T_{11} : W_a \rightarrow W_b$ and $W_b \rightarrow W_a$ gives a

proof of (4).

We begin our proofs by introducing elements P, E_4 , and B of $\mathbb{C}[[x]]$. These are the expansions at infinity of classical modular forms of level $\Gamma_0(5)$.

Definition 1.1 $P = 1 + \dots$, $E_4 = 1 + \dots$ and $B = x + \dots$ are the expansions at infinity of:

- (1) The normalized weight 2 Eisenstein series for $\Gamma_0(5)$.
- (2) The normalized weight 4 Eisenstein series of level 1.
- (3) The normalized weight 4 Eisenstein series, vanishing at infinity, for $\Gamma_0(5)$.

Remark One usually views expansions at infinity of modular forms as elements of $\mathbb{C}[[q]]$ where $q = e^{2\pi iz}$. But as in [1] we'll use the letter x rather than the letter q .

Definition 1.2 r in $Z/2[[x]]$ is $\sum_{n>0} (x^{n^2} + x^{2n^2} + x^{5n^2} + x^{10n^2})$.

Classical formulæ for the coefficients of Eisenstein series show that P , E_4 and B are in $Z[[x]]$ with mod 2 reductions 1, 1 and r .

Definition 1.3 C in $\mathbb{C}[[x]]$ is the expansion at infinity of the normalized weight 4 cusp form $(\eta(z)\eta(5z))^4$ for $\Gamma_0(5)$.

Now the expansion of $\eta(z)$ at infinity is $x^{1/24}$ (an element $1 - x - x^2 + \dots$ of $Z[[x]]$). We deduce:

Lemma 1.4 C is in $Z[[x]]$ and is $x - 4x^2 + \dots$.

We now show that the mod 2 reduction \bar{C} of C is $r^2 + r$.

Lemma 1.5 Let n be a positive integer. The number of (a, b) in $Z \times Z$ with $a^2 + 5b^2 = 6n$ and $a \equiv b \pmod{3}$ is 2 mod 4 if n is either a square or 5(square), and 0 mod 4 otherwise.

Proof Let $S(n)$ be the set of such (a, b) . Then $T : (a, b) \rightarrow (-a, -b)$ and $U : (a, b) \rightarrow \left(\frac{2a-5b}{3}, \frac{-a-2b}{3}\right)$ are commuting involutions of $S(n)$. T has no fixed points, while the fixed points of U (resp. TU) are of the form $(5k, -k)$ (resp. (k, k)). In the first case, $5k^2 = n$ while in the second $k^2 = n$. So we have an action of $Z/2 \times Z/2$ on $S(n)$ in which all orbits are of size 4 with the following exceptions. When $n = 5k^2$ there is a size 2 orbit $\pm(5k, -k)$. Where $n = k^2$ there is a size 2 orbit $\pm(k, k)$. The result follows. \square

Lemma 1.6 Let n be an integer. The number of (a, b) in $Z \times Z$ with $a^2 + 5b^2 = 6n$ and $a \equiv b \equiv 1 \pmod{6}$ is odd if n is an odd square or 5·(an odd square), and even otherwise.

Proof We may assume $n \geq 0$. Since $a^2 + 5b^2$ is 2 mod 4, there are no such pairs when n is even. For fixed odd $n > 0$ the pairs of Lemma 1.5 come in three types according as $a \equiv b \equiv 1 \pmod{6}$, $a \equiv b \equiv 5 \pmod{6}$, or $a \equiv b \equiv 3 \pmod{6}$. There are as many pairs of the first type as there are of the second. So in view of Lemma 1.5 it suffices to show that the number of pairs of the third type is a multiple of 4. But these pairs come in sets of 4, $(\pm a, \pm b)$. \square

Lemma 1.7 *The mod 2 reduction \bar{C} of C is $r^2 + r$.*

Proof Using the familiar expansion of $\eta(z)$ at infinity we find that $\bar{C} = \left(\sum_{a \equiv 1 \pmod{6}} x^{a^2/6}\right) \left(\sum_{b \equiv 1 \pmod{6}} x^{5b^2/6}\right)$. The coefficient of x^n in this element of $Z/2[[x]]$ is the mod 2 reduction of the number of (a, b) in $Z \times Z$ with $a^2 + 5b^2 = 6n$ and $a \equiv b \equiv 1 \pmod{6}$. So by Lemma 1.6, $\bar{C} = \sum_{n \text{ odd}, n > 0} (x^{n^2} + x^{5n^2})$. This is precisely $r^2 + r$. \square

Definition 1.8 *If $k \geq 0$ and even, M_k consists of those f in $Z/2[[x]]$ for which there is a weight k modular form of level $\Gamma_0(5)$ whose expansion at infinity lies in $Z[[x]]$ and reduces to f .*

Using multiplication by E_4 we see that $M_0 \subset M_4 \subset M_8 \subset \dots$

Definition 1.9 *$M = \cup M_{4m}$ is “the space of mod 2 modular forms of level $\Gamma_0(5)$.” $M(\text{odd})$ consists of the odd elements of M , i.e. those elements lying in $x \cdot Z/2[[x^2]]$.*

Note that M is a subring of $Z/2[[x]]$. Since the reductions of E_4 and B are 1 and r , $M \supset Z/2[r]$. Since $r^2 + r = \sum_{n \text{ odd}, n > 0} (x^{n^2} + x^{5n^2})$ is odd, each $r^{2k}(r^2 + r)$ is in $M(\text{odd})$.

Theorem 1.10 *Fix $m \geq 0$ and suppose $0 \leq i \leq 2m$. Then there is a weight $4m$ modular form u_i of level $\Gamma_0(5)$ whose expansion at infinity has the following properties. It is $x^i + \dots$, lies in $Z[[x]]$, and reduces to r^i .*

Proof It suffices to prove this when $m = 1$, so the weight is 4. For $i = 0$ we take P^2 whose expansion reduces to 1, while for $i = 1$ we take B whose expansion reduces to r . Now $E_4 = 1 + 240x + 2160x^2 + \dots$, $P^2 = (1 + 6x + 18x^2 + \dots)^2$ and $B = x + 9x^2 + \dots$ are expansions at infinity of weight 4 forms of level $\Gamma_0(5)$, and $E_4 - P^2 - 228B = 36x^2 + \dots$. Also $B - C = 13x^2 + \dots$, so a Z -linear combination of these last two expansions such as $4(\text{first}) - 11(\text{second})$ is $x^2 + \dots$. And the mod 2 reduction of this linear combination is $\bar{B} - \bar{C}$ which is r^2 by Lemma 1.7. \square

Theorem 1.11 *The r^i , $0 \leq i \leq 2m$, are a basis of M_{4m} over $Z/2$. It follows that $M = Z/2[r]$. Furthermore the $r^{2k}(r^2 + r)$ are a basis of $M(\text{odd})$ over $Z/2$.*

Proof Theorem 1.10 shows that the r^i , $0 \leq i \leq 2m$, lie in M_{4m} . Now the

u_i , $0 \leq i \leq 2m$, of Theorem 1.10 are linearly independent over \mathbb{C} . Classical dimension formulæ then show that over \mathbb{C} they span the space of weight $4m$ modular forms of level $\Gamma_0(5)$. Suppose u lies in this space and that the expansion of u at infinity lies in $Z[[x]]$. Writing u as a \mathbb{C} -linear combination of the u_i and examining the expansions we see that u is a Z -linear combination of the u_i . So the reduction of u is a $Z/2$ -linear combination of r^i , $0 \leq i \leq 2m$, giving the first two results. It follows also that the r^{2i} with $0 \leq i \leq m$ together with the $r^{2i}(r^2 + r)$ with $0 \leq i \leq m - 1$ are a $Z/2$ -basis of M_{4m} . We conclude that the $r^{2i}(r^2 + r)$ with $0 \leq i \leq m - 1$ are a basis for the subspace of M_{4m} consisting of odd power series. \square

Recall now that F in $Z/2[[x]]$ is $\sum_{n \text{ odd}, n > 0} x^{n^2}$, while $G = F(x^5)$. We have seen that $r^2 + r = F + G$.

Theorem 1.12

- (1) F and G lie in M_{12} .
- (2) $G = r^5(r + 1)$, $F = r(r + 1)^5$, and $(F + G)^6 = FG$.

Proof Let Δ be Ramanujan's weight 12 level 1 cusp form. Then F and G are the reductions of the expansions at infinity of $\Delta(z)$ and $\Delta(5z)$, and so lie in M_{12} . Theorem 1.11 shows that G is a $Z/2$ -linear combination of $r^2 + r$, $r^4 + r^3$ and $r^6 + r^5$. As $G = x^5 + \dots$ it can only be $r^6 + r^5$. Then $F = r^6 + r^5 + r^2 + r = r(r + 1)^5$. The final result is immediate. \square

Theorem 1.13 $M(\text{odd})$ is spanned by the $F^i G^j$ with $i + j$ odd.

Proof $r^2 + r = F + G$, while $r^4 + r^3 = (r^2 + r)^3 + r^6 + r^5 = (F + G)^3 + G$. Furthermore, $(r^4 + r^2)(r^{2k+2} + r^{2k+1}) + r^{2k+4} + r^{2k+3} = r^{2k+6} + r^{2k+5}$. Since $r^4 + r^2 = F^2 + G^2$, an induction on k shows that each $r^{2k+2} + r^{2k+1}$ is a sum of $F^i G^j$ with $i + j$ odd. \square

Suppose now that p is prime, $p \neq 2$ or 5 . Then we have commuting formal Hecke operators $T_p : Z/2[[x]] \rightarrow Z/2[[x]]$ with T_p taking $\sum c_n x^n$ to $\sum c_{pn} x^n + \sum c_n x^{pn}$.

Theorem 1.14 The T_p stabilize M and $M(\text{odd})$. In fact they stabilize the space spanned by the r^i with $0 \leq i \leq 2m$, as well as the space spanned by the $r^{2i}(r^2 + r)$, $0 \leq i \leq m - 1$.

Proof We may assume $m > 0$. Consider the Z -submodule of $Z[[x]]$ consisting of those elements of $Z[[x]]$ that are expansions at infinity of weight $4m$ modular form of level $\Gamma_0(5)$. This module is stabilized by each classical Hecke operator $T_p : \sum c_n x^n \rightarrow \sum c_{pn} x^n + p^{4m-1} \sum c_n x^{pn}$, $p \neq 5$. Reducing mod 2 and using Theorem 1.11 we get the result. \square

Remark Multiplication by G^2 stabilizes $M(\text{odd})$. Since $(F + G)^6 = FG$,

Theorem 1.13 shows that $M(\text{odd})$, viewed as $Z/2[G^2]$ -module, is spanned by F, F^3, F^5, G, F^2G and F^4G . As F has degree 6 over $Z/2(G)$, a basis of $M(\text{odd})$ as $Z/2[G^2]$ -module is $\{G, F, F^2G, F^3, F^4G, F^5\}$.

Definition 1.15

- (1) $N2 \subset M(\text{odd})$ has $Z/2[G^2]$ -basis $\{G, F, F^2G, F^3, F^4G\}$.
- (2) $N1 \subset N2$ has $Z/2[G^2]$ -basis $\{G\}$.

Definition 1.16 J_1, J_3, J_5, J_7 and J_9 are $F, F^8/G, G, F^2G$ and F^4G .

J_1, J_5, J_7 and J_9 are evidently in $N2$. Since $(F + G)^8 = FG(F + G)^2$, $F^8 = G^8 + FG(F + G)^2$, and $J_3 = G^7 + F(F + G)^2$ is also in $N2$. Now G, F, F^2G, F^3 and F^4G evidently generate the same $Z/2[G^2]$ -module as do J_5, J_1, J_7, J_3 and J_9 . Consequently:

Theorem 1.17

- (1) J_1, J_3, J_5, J_7 and J_9 are a $Z/2[G^2]$ -basis of $N2$, while J_1, J_3, J_7, J_9 are a $Z/2[G^2]$ -basis of $N2/N1$.
- (2) Define J_k , k odd, $k > 0$, by taking J_{k+10} to be G^2J_k . Then the J_k with $(k, 10) = 1$ are a $Z/2$ -basis of $N2/N1$.

Remarks

- (1) The space spanned by the F^k , k odd and > 0 , is stabilized by the T_p with $p \neq 2$. It follows that $N1$ is stabilized by the T_p with $p \neq 2$ or 5 .
- (2) One may describe $N2$ more elegantly as follows. $Z/2(F, G)$ is a degree 6 field extension of $Z/2(G)$, and we have a trace map, $Z/2(F, G) \rightarrow Z/2(G)$. Using the identity $(F + G)^6 = FG$ we find that F^i , $0 \leq i \leq 4$, have trace 0, while F^5 has trace G . So $N2$ consists of those elements of $M(\text{odd})$ of trace 0. We'll see in the next section that the T_p , $p \neq 2$ or 5 stabilize not only $M(\text{odd})$ and $N1$, but also $N2$.

2 The spaces W, W_a and W_b . A decomposition of $N2/N1$

Definition 2.1 $H = F(x^{25}) = G(x^5)$.

As we noted in section 1, $D = \sum_{(n,10)=1, n>0} x^{n^2}$ is $F + H$.

Definition 2.2 $pr : Z/2[[x]] \rightarrow Z/2[[x]]$ takes $\sum c_n x^n$ to $\sum_{(n,5)=1} c_n x^n$.

Since G lies in $Z/2[[x^5]]$, pr is $Z/2[G]$ -linear and $pr(N1) = 0$. The effect of pr on the $Z/2[G^2]$ -basis $\{J_1, J_3, J_7, J_9\}$ of $N2/N1$ is easily described:

Lemma 2.3 pr takes J_1, J_3, J_7, J_9 to $D, D^8/G, D^2G, D^4G$.

Proof $pr(J_3) = pr(F^8/G) = pr((H^8 + D^8)/G) = pr(D^8/G)$. Since all exponents appearing in D are prime to 5, the same holds for all exponents appearing in D^8/G and $pr(D^8/G) = D^8/G$. The other results have similar proofs. \square

Lemma 2.4 $D^{15} + G^4D^3 + G^3 = 0$.

Proof $(F + G)^6 = FG$. Replacing x by x^5 we find that $(G + H)^6 = GH$. So $(D + F + G)^6 = G(D + F)$. Adding FG to both sides, expanding in powers of D and dividing by D we find that $D^5 + (F + G)^2D^3 + (F + G)^4D = G$. So if we set $A = (F + G)^2$, then:

$$(1) \quad DA^2 + D^3A + (D^5 + G) = 0.$$

On the other hand, $A^3 = (F + G)^6 = FG$. So $A^6 = (A + G^2)G^2$, and:

$$(2) \quad A^6 + G^2A + G^4 = 0.$$

We can now eliminate A from (1) and (2) to get our result. Explicitly, when we multiply (1) by $DA^4 + D^3A^3 + GA^2 + D^7A$, (2) by D^2 , and add, all terms involving A^6 , A^5 , A^4 or A^3 drop out, and we find that $(D^{10} + D^5G + G^2)(A^2 + D^2A) + D^2G^4 = 0$. Let $B = A^2 + D^2A$. Then $(D^{10} + D^5G + G^2)B = D^2G^4$. Also, (1) tells us that $DB = D^5 + G$. So $(D^{10} + D^5G + G^2)(D^5 + G) = D^3G^4$, as desired. \square

Multiplying by D^5/G^4 gives:

Corollary 2.5 $(D^5/G)^4 + D^5/G = D^8$.

Definition 2.6 When $(i, 10) = 1$, $D_i = pr(J_i)$.

As we've seen, D_1, D_3, D_7 and D_9 are $D, D^8/G, D^2G$ and D^4G . And since pr is $Z/2[G^2]$ -linear, $D_{i+10} = G^2D_i$.

Theorem 2.7 The D_i are linearly independent over $Z/2$. So pr maps $N2/N1$ bijectively to the space W spanned by the D_i . (Recall that the J_k with $(k, 10) = 1$ are a $Z/2$ -basis of $N2/N1$.)

Proof G is transcendental over $Z/2$. It follows from Lemma 2.4 that D has degree 15 over $Z/2(G)$, so that D, D^8, D^2 and D^4 are linearly independent over this field. Consequently, $D_1 = D, D_3 = D^8/G, D_7 = D^2G$ and $D_9 = D^4G$ are linearly independent over $Z/2[G^2]$, giving the result. \square

Remark W does not consist of mod 2 modular forms of level 5. In fact the elements of W "are of level 25."

We shall see that the T_p ($p \neq 2$ or 5 as usual) stabilize both W and $N2$. To this end we use a real Dirichlet character χ , of modulus 20, to write W as

a direct sum of $Z/2[G^4]$ -submodules, W_a and W_b . Then we show that the T_p with $\chi(p) = 1$ stabilize W_a and W_b , while those with $\chi(p) = -1$ take W_a to W_b and W_b to W_a .

Definition 2.8 χ is the mod 20 Dirichlet character taking 1, 3, 7, 9 to 1, and taking 11, 13, 17, 19 to -1 .

Definition 2.9 W_a (resp. W_b) is the subspace of W spanned by the D_i with $\chi(i) = 1$ (resp. -1).

Evidently W_a and W_b are $Z/2[G^4]$ -submodules of W with bases $\{D_1, D_3, D_7, D_9\}$ and $\{D_{11}, D_{13}, D_{17}, D_{19}\}$. And $W = W_a \oplus W_b$.

Lemma 2.10 If x^n appears in D_k , n is k or $9k \bmod 40$. In particular when x^n appears in D_k , $\chi(n) = \chi(k)$.

Proof Since $D_{k+10} = G^2 D_k$ and all exponents appearing in G^2 are $10 \bmod 40$, it suffices to prove the lemma for k in $\{1, 3, 7, 9\}$. Suppose for example $k = 9$, so that $D_k = D^4 G$. If $(i, 10) = 1$, $i^2 \equiv 1$ or $9 \bmod 40$. So all exponents appearing in D are 1 or $9 \bmod 40$, and all exponents in $D^4 G$ are $4 + 5 = 9$ or $36 + 5 = 41 \bmod 40$. The other cases are handled similarly. \square

Definition 2.11

$$\begin{aligned} p_a : Z/2[[x]] &\rightarrow Z/2[[x]] \text{ takes } \sum c_n x^n \text{ to } \sum_{\chi(n)=1} c_n x^n. \\ p_b : Z/2[[x]] &\rightarrow Z/2[[x]] \text{ takes } \sum c_n x^n \text{ to } \sum_{\chi(n)=-1} c_n x^n. \end{aligned}$$

p_a and p_b are evidently $Z/2[G^4]$ -linear. Furthermore:

Lemma 2.12 $p_a(G^2 f)$ and $p_b(G^2 f)$ are $G^2 p_b(f)$ and $G^2 p_a(f)$.

Proof We may assume that all the exponents appearing in f are congruent to some fixed $k \bmod 20$. Then the exponents in $G^2 f$ are congruent to $k + 10$, and we use the fact that $\chi(k + 10) = -\chi(k)$. \square

Lemma 2.13

$$\begin{aligned} \text{If } \chi(k) = 1, & \ p_a(J_k) = D_k \text{ and } p_b(J_k) = 0. \\ \text{If } \chi(k) = -1, & \ p_a(J_k) = 0 \text{ and } p_b(J_k) = D_k. \end{aligned}$$

Proof Since $pr(J_k) = D_k$, $p_a(J_k) = p_a(D_k)$ while $p_b(J_k) = p_b(D_k)$. Lemma 2.10 then gives the result. \square

As $Z/2[G^4]$ -module, $N2/N1$ has basis $\{J_1, J_3, J_7, J_9, J_{11}, J_{13}, J_{17}, J_{19}\}$. Evidently $N2/N1 = N2a \oplus N2b$, where $N2a$ and $N2b$ are the $Z/2[G^4]$ -submodules with bases $\{J_1, J_3, J_7, J_9\}$ and $\{J_{11}, J_{13}, J_{17}, J_{19}\}$. The J_k with $\chi(k) = 1$ are a $Z/2$ -basis of $N2a$; those with $\chi(k) = -1$ are a $Z/2$ -basis of $N2b$. Lemma 2.13 now shows that p_a maps $N2/N1$ onto W_a with kernel $N2b$, while p_b maps

$N2/N1$ onto W_b with kernel $N2a$.

Suppose now that f is in $M(\text{odd})/N1$. We'll show that if $p_b(f) = 0$, then f is in $N2a$, while if $p_a(f) = 0$, f is in $N2b$. So in either case, f is in $N2/N1$. This is key to showing that the T_p stabilize $N2$.

Lemma 2.14 $p_a(F(F+G)^4) = p_a(r^8G)$ and $p_b(F(F+G)^4) = p_b(r^8G)$.

Proof $F(F+G)^4 = r(r+1)^5(r^2+r)^4 = (r+1)^8(r^6+r^5) = r^8G + G$. Now apply p_a and p_b . \square

Lemma 2.15 Let $S = p_a(r^8G)$, $T = p_b(r^8G)$. Then $T = D^5 + G$, $S = D^{10}/G + G$.

Proof Since $r+r^2 = F+G$, $r^8G = G((F+G)^8 + (F+G)^{16} + (F+G)^{32} + \dots)$. So $pr(r^8G) = G(D^8 + D^{16} + D^{32} + \dots)$. Now all exponents n appearing in any of GD^8 , GD^{32} , GD^{128} , ... are 13 or 37 mod 40, while those in any of GD^{16} , GD^{64} , ... are 21 or 69 mod 40. Applying p_b to our identity we find that $T = G(D^8 + D^{32} + D^{128} + \dots)$. Then $(T/G)^4 + (T/G) = D^8$. So by Corollary 2.5, T/G and D^5/G differ by a constant, and comparing expansions we see that the constant is 1. So $T = D^5 + G$. Finally $S = G(D^{16} + D^{64} + D^{256} + \dots) = T^2/G$. \square

Lemma 2.16

- (1) $p_a(F(F+G)^4) = D^{10}/G + G$, $p_b(F(F+G)^4) = D^5 + G$.
- (2) $p_a(FG^2(F+G)^4) = D^5G^2 + G^3$, $p_b(FG^2(F+G)^4) = D^{10}G + G^3$.

Proof Lemmas 2.14 and 2.15 give (1), and Lemma 2.12 yields (2). \square

We saw in the remark following the proof of Theorem 1.14 that a basis of $M(\text{odd})$ as $Z/2[G^2]$ -module is $\{G, F, F^2G, F^3, F^4G, F^5\}$. The last five of those elements then give a basis of $M(\text{odd})/N1$. It follows that another $Z/2[G^2]$ -basis of $M(\text{odd})/N1$ is $\{J_1, J_3, J_7, J_9, F(F+G)^4\}$. Then a $Z/2[G^4]$ -basis of $M(\text{odd})/N1$ is $\{J_1, J_3, J_7, J_9, J_{11}, J_{13}, J_{17}, J_{19}, F(F+G)^4, FG^2(F+G)^4\}$.

Theorem 2.17 The kernels of $p_b : M(\text{odd})/N1 \rightarrow Z/2[[x]]$ and $p_a : M(\text{odd})/N1 \rightarrow Z/2[[x]]$ are $N2a$ and $N2b$ where $N2a$ and $N2b$ have $Z/2[G^4]$ -module bases $\{J_1, J_3, J_7, J_9\}$ and $\{J_{11}, J_{13}, J_{17}, J_{19}\}$.

Proof Consider the 10 element $Z/2[G^4]$ -module basis of $M(\text{odd})/N1$ given in the sentence preceding Theorem 2.17. p_b annihilates J_1, J_3, J_7 and J_9 and sends the last 6 basis elements to $D_{11} = DG^2$, $D_{13} = D^8G$, $D_{17} = D^2G^3$, $D_{19} = D^4G^3$, $D^5 + G$ and $D^{10}G + G^3$; see Lemma 2.16. As we've seen, D has degree 15 over $Z/2(G)$, and so $D, D^8, D^2, D^4, D^5 + G$ and $D^{10} + G^2$ are linearly independent over this field. So no non-trivial $Z/2[G^4]$ -linear combination of $D_{11}, D_{13}, D_{17}, D_{19}, D^5 + G$ and $D^{10} + G^3$ is zero, and the result for p_b follows.

The result for p_a is proved similarly. \square

Corollary 2.18

- (1) If $\chi(p) = 1$, T_p stabilizes $N2a$ and $N2b$.
- (2) If $\chi(p) = -1$, T_p maps $N2a$ to $N2b$ and $N2b$ to $N2a$.

Proof For example suppose f is in $N2b$ with $\chi(p) = -1$. Then the exponents, n , appearing in (a representative of) f have $\chi(n) = -1$ or 0 . Since χ is multiplicative, the exponents appearing in T_p applied to this representative have $\chi(n) = 1$ or 0 . In other words, $T_p(f)$ is an element of $M(\text{odd})/N1$ in the kernel of p_b . By Theorem 2.17, $T_p(f)$ is in $N2a$. The other cases are treated similarly. \square

Theorem 2.19 If $p \neq 2$ or 5 , T_p stabilizes $N2$ and W .

Proof If h is in $N2$, the image of h in $N2/N1$ is the sum of an element of $N2a$ and an element of $N2b$. By Corollary 2.18, the same holds for the image of $T_p(h)$ in $M(\text{odd})/N1$. So this image is in $N2/N1$ and h is in $N2$. Suppose f is in W . Then $f = pr(h)$ with h in $N2$. Then $T_p(h)$ is in $N2$, and $T_p(f) = pr(T_p(h))$ is in W . \square

Corollary 2.20 Suppose $p \neq 2$ or 5 , and $(n, 10) = 1$. Then $T_p(D_n)$ is a sum of distinct D_k . In such a decomposition each k is either pn or $9pn \bmod 40$, and in particular $\chi(k) = \chi(p)\chi(n)$.

Proof Since D_n is in W , so is $T_p(D_n)$, giving the first result. Also the exponents appearing in $T_p(D_n)$ are all congruent to pn or $9pn \bmod 40$ (by Lemma 2.10 and the definition of T_p) while those appearing in a D_k are k or $9k \bmod 40$. It follows easily that those D_k appearing in the sum for which k is neither congruent to pn nor to $9pn \bmod 40$ sum to 0 . So there are no such k . \square

Remark Since $N2/N1 = N2a \oplus N2b$, $pr: N2/N1 \rightarrow Z/2[[x]]$ is 1-1 with image $W_a \oplus W_b = W$. So pr gives an identification of the subquotient $N2/N1$ of $M(\text{odd})$ with W . Now the T_p , $p \neq 2$ or 5 , stabilize $N2/N1$ and W , and the identification preserves the action of T_p . Corollary 2.20 and this identification are the quoted results (1) and (2) at the beginning of section 1.

We show next that when $p \equiv 3$ or $7 \pmod{10}$, each k in Corollary 2.20 is $< n$. (This is also true when $p \equiv 1$ or 9 , but this will only be proved in the final section.)

Definition 2.21 $v_2 = r^2 + r$, $v_4 = r^4 + r^3$, $v_6 = r^6 + r^5$, $v_{10} = r^{10} + r^9 + r^8 + r^7$, $v_{12} = r^{12} + r^{11} + r^{10} + r^9$.

Note that $v_2 = F + G$, $v_4 = (F + G)^3 + G$, $v_6 = G$, $v_{10} = (F + G)^2 G$ and $v_{12} = (F + G)^4 G + (F + G)G^2$. So v_2 , v_4 , v_6 , v_{10} and v_{12} generate the same

$Z/2[G^2]$ -submodule of $M(\text{odd})$ as do G, F, F^2G, F^3 and F^4G ; that is to say they generate $N2$. Since $G^2 = r^{12} + r^{10}$, $G^{2s}v_j$ is an element of $N2$ whose degree in r is $12s + j$.

Lemma 2.22 $T_p(D_{10m+3})$ and $T_p(D_{10m+7})$ are sums of D_k with $k \leq 10m + 7$.

Proof $J_7 = F^2G \equiv v_{10} \pmod{N1}$. And $J_3 = F^8/G = (F+G)^8/G + G^7 = F(F+G)^2 + G^7$. So $\pmod{N1}$, $J_3 + J_7 \equiv (F+G)^3 \equiv v_4$. It follows that $J_{10m+3} = G^{2m}J_3$ and $J_{10m+7} = G^{2m}J_7$ are each congruent $\pmod{N1}$ to polynomials in r of degree $\leq 12m + 10$. By Theorem 1.14 the same is true of $T_p(J_{10m+3})$ and $T_p(J_{10m+7})$. (Note also that $J_1 = F \equiv v_2 \pmod{N1}$ while $J_9 + J_{11} = F^4G + FG^2 \equiv v_{12}$.) Now take an h of degree $\leq 12m + 10$ in r which is congruent $\pmod{N1}$ to $T_p(J_{10m+3})$ (or to $T_p(J_{10m+7})$). Since J_{10m+3} and J_{10m+7} are in $N2$, so is h . Write h as a sum of distinct $G^{2s}v_i$ with each i in $\{2, 4, 6, 10, 12\}$. The degree in r of $G^{2s}v_i$ is $12s + i$. These degrees are distinct, and since the degree of h is $\leq 12m + 10$, the s appearing in those $G^{2s}v_i$ with $i = 12$ are all $< m$, while the remaining s are all $\leq m$. Since v_2, v_4, v_6, v_{10} and v_{12} are congruent $\pmod{N1}$ to $J_1, J_3 + J_7, 0, J_7$ and $J_9 + J_{11}$, and each of $10m + 1, 10m + 7, 10m + 7, 10(m - 1) + 11$ is $\leq 10m + 7$, each $G^{2s}v_i$ appearing in the sum for h is, $\pmod{N1}$, a sum of J_k with $k \leq 10m + 7$. Applying pr to the identity $h = \sum G^{2s}v_i$, noting that pr preserves the action of T_p and that $pr(J_k)$ is either D_k or 0, we get the result. \square

Lemma 2.23 $T_p(D_{10m+9})$ and $T_p(D_{10m+11})$ are sums of D_k with $k \leq 10m + 11$.

Proof $J_{11} = G^2J_1 \equiv G^2v_2 \pmod{N1}$, while $J_9 + J_{11} \equiv v_{12}$. It follows that J_{10m+9} and J_{10m+11} are congruent $\pmod{N1}$ to polynomials in r of degree $\leq 12m + 14$. By Theorem 1.14, the same is true of $T_p(J_{10m+9})$ and $T_p(J_{10m+11})$. Now take an h of degree $\leq 12m + 14$ in r which $\pmod{N1}$ is $T_p(J_{10m+9})$ (or $T_p(J_{10m+11})$). h is in $N2$ and we write it as a sum of distinct $G^{2s}v_i$, i in $\{2, 4, 6, 10, 12\}$. Arguing as in the proof of Lemma 2.22 we find that the s appearing in the $G^{2s}v_i$ with $i = 2$ are $\leq m + 1$, while the remaining s are $\leq m$, and we continue as in the proof of Lemma 2.22. \square

Theorem 2.24 Suppose $p \equiv 3$ or $7 \pmod{10}$. When we write $T_p(D_n)$ as a sum of distinct D_k , each $k < n$.

Proof Suppose $n = 10m + 3$ or $10m + 7$. By Lemma 2.22, $T_p(D_n)$ is a sum of distinct D_k , $k \leq 10m + 7$. By Corollary 2.20, each $k \equiv pn$ or $9pn \pmod{10}$ and so is 1 or 9 $\pmod{10}$. So no k can be $10m + 3$ or $10m + 7$. If $n = 10m + 9$ or $10m + 11$, $T_p(D_n)$ is, by Lemma 2.23, a sum of D_k , $k \leq 10m + 11$. Then each $k \equiv pn$ or $9pn \pmod{10}$ and so is 3 or 7 $\pmod{10}$. So no k is $10m + 9$ or $10m + 11$. Finally, $T_p(D_1) = T_p(D) = 0$. \square

We shall write down a linear recursion satisfied by the $T_3(D_n)$, $n \equiv 1, 3, 7, 9 \pmod{20}$.

This recursion, together with some initial condition results, proved with the help of Theorem 2.24 when $p = 3$, allow us to relate $T_3(D_n)$ to a polynomial P_n appearing in section 5 of [2].

Lemma 2.25 *For u in $Z/2[[x]]$, $T_3(G^{16}u) = G^{16}T_3(u) + G^4T_3(G^4u)$.*

Proof Let u be the 2-variable polynomial $A^4 + B^4 + AB$. We have the “level 3 modular equation for F ,” $U(F(x^3), F(x)) = 0$. Replacing x by x^5 shows that $U(G(x^3), G(x)) = 0$. Now proceed as in the proof of Lemma 2.19 of [1] (though now we have 4 imbeddings φ_i , the first taking $f(x)$ to $f(x^3)$, and the others taking $f(x)$ to $f(\lambda x^{1/3})$ where λ runs over the cube roots of 1 in an algebraic closure of $Z/2$). \square

Theorem 2.26 *The $T_3(D_n)$. $n \equiv 1, 3, 7, 9 \pmod{20}$ satisfy the recursion $T_3(D_{n+80}) = G^{16}T_3(D_n) + G^4T_3(D_{n+20})$. And T_3 takes*

$$D_1, D_3, D_7, D_9, D_{21}, D_{23}, D_{27}, D_{29}, D_{41}, D_{43}, D_{47}, D_{49}, D_{61}, D_{63}, D_{67}, D_{69}$$

to:

$$\begin{aligned} &0, D_1, 0, D_3, D_7, D_{21}, D_9, D_{23}, 0, D_{41}, D_{21}, D_{43} + D_{27}, \\ &D_{47} + D_{23}, D_{61} + D_{29} + D_{21}, D_{49} + D_{41}, D_{63} + D_{47} + D_{23}. \end{aligned}$$

Proof Sketch Taking $u = D_n$ in Lemma 2.25, we get the recursion. I’ll calculate $T_3(D_{47})$ explicitly, the other 15 initial values being derived in a similar way. By Corollary 2.20 and Theorem 2.24, $T_3(D_{47})$ is a $Z/2$ -linear combination of D_{21} and D_{29} . Now $D_{47} = G^8(D^2G) = (x^{40} + x^{360} + \dots)(x^2 + x^{18} + x^{98} + \dots)(x^5 + x^{45} + \dots)$. So the coefficients of x^7 , x^{63} and x^{87} in D_{47} are 0, 1 and 1. It follows that the coefficients of x^{21} and x^{29} in $T_3(D_{47})$ are each 1. But $D_{21} = G^4D = x^{21} + x^{29} + \dots$ while $D_{29} = x^{29} + \dots$. It follows that $T_3(D_{47}) = D_{21}$. \square

Now let w be an indeterminate over $Z/2$.

Definition 2.27 $V' \subset Z/2[w]$ is the space spanned by the w^k with $k \equiv 1, 3, 7, 9 \pmod{20}$.

There is a $Z/2$ -linear identification of W_a with V' taking D_k to w^k . Then $T_3 : W_a \rightarrow W_a$ goes over to a map $V' \rightarrow V'$ which we’ll still call T_3 . Now in section 5 of [2] (see Theorem 5.1, the paragraphs preceding it, and Remark 5.2) we defined certain P_k , $k \equiv 1, 3, 7, 9 \pmod{20}$ in V' .

Theorem 2.28 $T_3 : V' \rightarrow V'$ takes w^k to the P_k of [2] section 5.

Proof Let $A_k = T_3(w^k)$. The recursion of Theorem 2.26 tells us that $A_{k+80} = w^{80}A_k + w^{20}A_{k+20}$, while Theorem 5.1 of [2] tells us that $P_{k+80} = w^{80}P_k + w^{20}P_{k+20}$. Putting the initial values in Theorem 2.26 together with Theorem

5.1 of [2] we see that $P_k = A_k$ whenever $k < 80$, $k \equiv 1, 3, 7, 9 \pmod{20}$. The recursions then show that $P_k = A_k$ whenever $k \equiv 1, 3, 7, 9 \pmod{20}$. \square

3 Coding a basis of W_a . The effect of T_3 on the code

Define a total ordering on $N \times N$ as follows. (c, d) “precedes” (or “is earlier than”) (a, b) if $c + d < a + b$ or $c + d = a + b$ and $d < b$.

Consider the space W_a with its basis $\{D_k, k \equiv 1, 3, 7, 9 \pmod{20}\}$. In this section we “code the basis” by identifying it with $N \times N$; $(c, d)^*$ will denote the particular D_k corresponding to (c, d) . Under this identification, the total ordering on $N \times N$ given in the last paragraph goes over to a total ordering on our basis, and we use the “precedence” language of the last paragraph and show:

- (1) $T_3(a, b)^*$ is a sum of $(c, d)^*$ with $c + d < a + b$.
- (2) If $a > 0$, $T_3(a, b)^* = (a - 1, b)^* +$ a sum of earlier D_j .

The hard work in establishing (1) and (2) has already been done in Lemma 5.5 of [2] and Theorem 2.28 of the last section, so most of this section is a summary of results from [2].

Definition 3.1 $g : N \rightarrow N$ is the function with $g(2n) = 4g(n)$ and $g(2n + 1) = g(2n) + 1$.

Definition 3.2 $V \subset Z/2[t]$ is spanned by the t^k , k odd. If (a, b) is in $N \times N$, $[a, b]$ in V is $t^{1+2g(a)+4g(b)}$.

$(a, b) \rightarrow [a, b]$ sets up a 1–1 correspondence between $N \times N$ and the monomial basis of V . The total ordering of $N \times N$ goes over to a total ordering of the basis. Once again we use the language of “precedence.” We now pass from V to V' .

Definition 3.3 If V' is as in Definition 2.27, $\varphi : V \rightarrow V'$ is the $Z/2$ -linear bijection with:

$$\begin{aligned} \varphi(t^{16m+1}) &= w^{40m+1}, & \varphi(t^{16m+3}) &= w^{40m+3} \\ \varphi(t^{16m+5}) &= w^{40m+7}, & \varphi(t^{16m+7}) &= w^{40m+21} \\ \varphi(t^{16m+9}) &= w^{40m+9}, & \varphi(t^{16m+11}) &= w^{40m+27} \\ \varphi(t^{16m+13}) &= w^{40m+23}, & \varphi(t^{16m+15}) &= w^{40m+29} \end{aligned}$$

Definition 3.4 $\langle a, b \rangle$ in V' is $\varphi[a, b]$. $(a, b)^*$ in W_a is the image of $\langle a, b \rangle$ under the identification of V' with W_a taking w^k to D_k .

Note that $\varphi : V \rightarrow V'$ identifies the monomial basis (in t) of V with the

monomial basis (in w) of V' . So Definition 3.4 results in a coding of the basis $\{D_k\}$ of W_a .

Theorem 3.5

- (1) $T_3(a, b)^*$ is a sum of $(c, d)^*$ with $c + d < a + b$.
- (2) If $a > 0$, $T_3(a, b)^* = (a - 1, b)^* +$ a sum of earlier D_j .

Proof We saw in the last section that T_3 stabilizes W_a . In view of the identification of W_a with V' it suffices to show that $T_3\langle a, b \rangle$ is a sum of $\langle c, d \rangle$ with $c + d < a + b$, and that if $a > 0$, $T_3\langle a, b \rangle = \langle a - 1, b \rangle +$ a sum of earlier monomials. Suppose then that $\langle a, b \rangle = w^k$. By Theorem 2.28, $T_3\langle a, b \rangle$ is the P_k of [2], Theorem 5.1. So our result is precisely Lemma 5.5 of [2]. \square

We will need one further property of our code.

Theorem 3.6 *If $D_i = (c, d)^*$ precedes $D_j = (0, b)^*$, then $i < j$.*

Proof In view of our identification of W_a with V' , it's enough to show that if $w^i = \langle c, d \rangle$ precedes $w^j = \langle 0, b \rangle$, then $i < j$.

Now there is the following corresponding result for V . If $t^i = [c, d]$ precedes $t^j = [0, b]$ then $i < j$. For $c + d \leq b$, so by Lemma 4.1 of [1], $4g(c) + 4g(d) \leq 4g(b)$. Then $i = 1 + 2g(c) + 4g(d) \leq 1 + 4g(b) = j$, and since $i \neq j$, $i < j$.

We'll deduce Theorem 3.6 from this result for V . Suppose first that b is even, and let $m = g(b/2)$. Then $1 + 4g(b) = 16m + 1$, and $[0, b] = t^{16m+1}$. Write $[c, d]$ as $t^{16m'+r}$ with r in $\{1, 3, 5, 7, 9, 11, 13, 15\}$. Since $\langle c, d \rangle$ precedes $\langle 0, b \rangle$, $[c, d]$ precedes $[0, b]$. By the result of the last paragraph, $16m' + r < 16m + 1$ and so $m' < m$. Now $\langle 0, b \rangle = w^{40m+1}$ so that $j = 40m + 1$. Similarly, $i \leq 40m' + 29 \leq 40m - 11$, and this is $< j$. The argument when b is odd is similar. Let $m = g((b-1)/2)$, so that $g(b) = 4m + 1$, and $[0, b] = t^{16m+5}$. Let m' and r be as in the case of even b . Arguing as in the case of even b we find that $16m' + r < 16m + 5$. If $m' < m$ we proceed as in the case of even b . If $m' = m$ then r must be 1 or 3. So $\langle c, d \rangle$ is either w^{40m+1} or w^{40m+3} , while $\langle 0, b \rangle = \varphi(t^{16m+5}) = w^{40m+7}$, and once again $i < j$. \square

4 Type a ideals of $Z[\sqrt{-10}]$ and Gauss-classes

This section is the counterpart to section 3 of [1]. Fix a power, q , of 2. We shall (essentially) use binary quadratic forms of discriminant $-640q^2$ and their associated theta-series to construct a subspace $DI(q)$ of W_a of dimension $2q$, stable under the T_p with $\chi(p) = 1$, and annihilated by T_3 . and we'll give a simple description of the action of T_7 on $DI(q)$ involving Gaussian composition of

forms. We will not use all primitive positive forms of discriminant $-640q^2$; the Dirichlet character χ of Definition 2.8 may be thought of as a genus character, and we'll only consider forms on which this character takes the value 1. We'll see that the $SL_2(Z)$ -classes of such forms make up a cyclic group of order $4q$, and that the class of a form representing 7 is a generator.

As in section 3 of [1], we'll avoid the explicit language of binary forms. Instead we'll consider ideals I in $Z[\sqrt{-10}]$ for which $\chi(\text{norm}(I)) = 1$. We'll say that such an ideal is "of type a ." We fix a power q of 2, and introduce an equivalence relation, depending on q , on the type a ideals. We will call our equivalence relation "Gauss-equivalence," and the equivalence classes under it "Gauss-classes."

The class number of $Z[\sqrt{-10}]$ is 2, with an ideal of norm 7 representing the non-principal class. We begin by defining Gauss-equivalence for principal ideals $(b + c\sqrt{-10})$ of type a .

Since the norm of $(b + c\sqrt{-10})$ is $b^2 + 10c^2$, $(b + c\sqrt{-10})$ is of type a precisely when $(b, 10) = 1$ and c is even. Note also that the generator of a principal ideal is defined up to multiplication by ± 1 .

Definition 4.1 *Principal type a ideals (α) and (β) are equivalent if there is an integer N with $(N, 10) = 1$ such that $N\alpha \equiv \beta \pmod{4q}$ in the ring $Z[\sqrt{-10}]$.*

Evidently this does not depend on the choices of generators for the ideals, and is an equivalence relation. Also ideal multiplication makes the set of equivalence classes into a semigroup. Since $(\alpha)(\bar{\alpha}) = (\text{norm}(\alpha))$ which is equivalent to (1) , the semigroup is a group.

Lemma 4.2

- (1) *Any principal type a ideal (α) is equivalent to $(1 + 2d\sqrt{-10})$ for some d .*
- (2) *$(1 + 2c\sqrt{-10})$ and $(1 + 2d\sqrt{-10})$ are equivalent if and only if $c \equiv d \pmod{2q}$.
So there are $2q$ equivalence classes.*

Proof Write α as $(b + 2c\sqrt{-10})$. Then $(b, 10) = 1$ and we can choose N with $(N, 10) = 1$ so that $Nb \equiv 1 \pmod{4q}$. Then (Na) is of type a , and $N\alpha \equiv 1 + 2Nc\sqrt{-10} \pmod{4q}$, proving (1). Turning to (2), if $c \equiv d \pmod{2q}$ we may take $N = 1$. Conversely if $N(1 + 2c\sqrt{-10}) \equiv 1 + 2d\sqrt{-10} \pmod{4q}$, then $N \equiv 1 \pmod{4q}$. So $\pmod{4q}$, $2d \equiv 2Nc \equiv 2c$. \square

Theorem 4.3 *The order $2q$ group of classes of principal type a ideals is cyclic, generated by the class of any $(c + 2d\sqrt{-10})$ with $(c, 10) = 1$ and d odd.*

Proof One argues as in the last few sentences of the proof of Theorem 3.2 of [1]. \square

Now fix a type a non-principal ideal L ; for example $P = (7, 2 - \sqrt{-10})$. If I is another such ideal, IL is principal of type a .

Definition 4.4 Suppose I and J are type a non-principal. I and J are Gauss equivalent if IL and JL are equivalent in the sense of Definition 4.1.

The definition appears to depend on the choice of L . But if one replaces L by γL where $\chi(\text{norm}(\gamma)) = 1$, one finds that the notion of Gauss-equivalence is unchanged. It follows at once that the dependence on L is illusory. One sees further that the Gauss-classes of all type a ideals form a group of order $4q$ under ideal multiplication and that the inverse of I is \bar{I} .

Theorem 4.5 The group of Gauss-classes of type a ideals is cyclic of order $4q$; any class consisting of non-principal ideals is a generator.

Proof Let $P = (7, 2 - \sqrt{-10})$. P has norm 7, and is of type a . Also, $P^2 = (3 + 2\sqrt{-10})$. By Theorem 4.3, P^2 has order $2q$. So P has order $4q$, and the result follows. \square

Definition 4.6 If R is a Gauss-class, $\theta(R)$ in $Z[[x]]$ is $\sum x^{\text{norm}(I)}$, where I runs over the ideals in R .

Definition 4.7

- (1) e is the Gauss-class of (1).
- (2) AMB is the Gauss-class of order 2; that is to say the Gauss-class of $(1 + 2q\sqrt{-10})$.

Lemma 4.8

- (1) The mod 2 reduction of $\theta(e)$ is D .
- (2) $\theta(AMB)$ is in $2Z[[x]]$, and the mod 2 reduction of $\frac{1}{2}\theta(AMB)$ is D_{40q^2+1} .

Proof I is in e if and only if \bar{I} is in e ; also they each make the same contribution to $\theta(e)$. So in calculating $\theta(e) \bmod 2$ we only need consider I in e with $\bar{I} = I$. These are just the (N) with N in Z , $(N, 10) = 1$ and $N > 0$, and we get (1). AMB consists of principal ideals. These are of the form $(b + 2cq\sqrt{-10})$ with $(b, 10) = 1$, $b > 0$ and c odd. Since $(b + 2cq\sqrt{-10})$ and $(b - 2cq\sqrt{-10})$ make the same contribution to $\theta(AMB)$, $\theta(AMB)$ is in $2Z[[x]]$. Also, $\frac{1}{2}\theta(AMB)$ is $\sum x^{b^2+40q^2c^2}$, the sum running over all positive b and c with $(b, 10) = 1$ and c odd. Reducing mod 2 we get $DG^{8q^2} = D_{40q^2+1}$. \square

We next define a Hecke operator $T_p : Z[[x]] \rightarrow Z[[x]]$ for each p with $\chi(p) = 1$.

Definition 4.9 If $\chi(p) = 1$, T_p is the map $\sum c_n x^n \rightarrow \sum c_{pn} x^n + \left(\frac{-10}{p}\right) \sum c_n x^{pn}$.

Note that the mod 2 reduction of T_p is our usual mod 2 Hecke operator $T_p : Z/2[[x]] \rightarrow Z/2[[x]]$.

Remark The motivation for Definition 4.9 is the following. It can be shown that each $\theta(R)$ is the expansion at infinity of a modular form of weight 1 for some $\Gamma_1(N)$, with character $n \rightarrow \left(\frac{-10}{n}\right)$. And Definition 4.9 is the standard definition of the Hecke action on such expansions. (But in what follows we won't use the connection of the $\theta(R)$ with weight 1 modular forms.)

Theorem 4.10 Suppose $\chi(p) = 1$ and p is inert in $Z[\sqrt{-10}]$. Then T_p annihilates each $\theta(R)$ in $Z[[x]]$.

Proof $I \rightarrow pI$ sets up a 1-1 correspondence between ideals of norm n in R and ideals of norm $p^2 n$ in R . Also, if $(n, p) = 1$ there are no ideals of norm pn in R . Since $\left(\frac{-10}{n}\right) = -1$, the result follows directly from Definition 4.9. \square

Theorem 4.11 Suppose $\chi(p) = 1$ and p splits in $Z[\sqrt{-10}]$, so that $(p) = P \cdot \bar{P}$ with $P \neq \bar{P}$. Then if R is a Gauss-class, $T_p(\theta(R)) = \theta(PR) + \theta(\bar{P}R)$. (Since $\chi(p) = 1$, P and \bar{P} are type a .)

Proof The argument follows that in the proof of Theorem 3.6 of [1], the essential points being the multiplicativity of the norm and unique factorization at the ideal level in $Z[\sqrt{-10}]$. \square

Definition 4.12 $\alpha(R)$ is the mod 2 reduction of $\theta(R)$. $DI(q) \subset Z/2[[x]]$ is the space spanned by the $\alpha(R)$, as R runs over the $4q$ Gauss-classes of type a ideals.

Now let C be a Gauss-class containing one of the ideals of norm 7. By Theorem 4.5, C generates the group of Gauss-classes of type a ideals.

Theorem 4.13 Let $\alpha_i = \alpha(C^i)$. Then $\alpha_0 = D$, $\alpha_{2q} = 0$, and the α_i , $0 \leq i < 2q$ span $DI(q)$.

Proof Since C is a generator, $\alpha_0, \dots, \alpha_{4q-1}$ span $DI(q)$. By Lemma 4.8, $\alpha_0 = D$ and $\alpha_{2q} = 0$. Finally $I \rightarrow \bar{I}$ sets up a 1-1 norm preserving correspondence between the ideals in C^i and the ideals in C^{4q-i} , so $\alpha_i = \alpha_{4q-i}$. \square

Theorem 4.14

- (1) $T_7(\alpha_i) = \alpha_{i-1} + \alpha_{i+1}$ if $0 < i < 2q$. So T_7 stabilizes $DI(q)$.
- (2) $T_7(D_{40q^2+1}) = \alpha_{2q-1}$.

Proof By Theorem 4.11, $T_7(\theta_i) = \theta_{i-1} + \theta_{i+1}$; reducing mod 2 we get (1). Also $T_7(\theta(AMB)) = \theta_{2q-1} + \theta_{2q+1} = 2\theta_{2q-1}$. Dividing by 2, reducing mod 2

and using Lemma 4.8 we get (2). \square

Definition 4.15 U_n is the element of $Z/2[t]$ with $U_n(t+t^{-1}) = t^n + t^{-n}$. Note that $U_0 = 0$, $U_1(Y) = Y$, and that $U_{n+2}(Y) = YU_{n+1}(Y) + U_n(Y)$. Furthermore $U_{2n} = U_n^2$, and it follows that $U_q(Y) = Y^q$. Finally Y divides each $U_n(Y)$.

Lemma 4.16 Let Y be the operator $T_7 : Z/2[[x]] \rightarrow Z/2[[x]]$. Then for $0 \leq i \leq 2q$, $\alpha_{2q-i} = U_i(Y) \cdot (D_{40q^2+1})$.

Proof If $i \leq 2q - 2$, Theorem 4.14 and the recursion in Definition 4.15 give:

- (1) $\alpha_{2q-i-2} = Y(\alpha_{2q-i-1}) + \alpha_{2q-i}$.
- (2) $U_{i+2}(Y) = Y \cdot U_{i+1}(Y) + U_i(Y)$.

So, by induction on i , it suffices to show that α_{2q} and α_{2q-1} are $U_0(Y)(D_{40q^2+1})$ and $U_1(Y)(D_{40q^2+1})$. But Lemma 4.8 and Theorem 4.14 show that $\alpha_{2q} = 0$ and $\alpha_{2q-1} = Y \cdot D_{40q^2+1}$. \square

Theorem 4.17 As $Z/2[Y]$ -module, $DI(q)$ is cyclic with generator $\alpha_{2q-1} = Y \cdot D_{40q^2+1}$.

Proof $\alpha_{2q-i} = U_i(Y) \cdot D_{40q^2+1}$ for $1 \leq i \leq 2q$. Since Y divides each $U_i(Y)$ we're done. \square

Theorem 4.18 $Y^{2q-1}(\alpha_{2q-1}) = D$, while $Y^{2q}(\alpha_{2q-1}) = 0$. Also, $DI(q)$ has dimension $2q$ over $Z/2$ and is isomorphic as $Z/2[Y]$ -module with $Z/2[Y]/Y^{2q}$.

Proof $Y^{2q-1}(\alpha_{2q-1}) = Y^{2q}(D_{40q^2+1}) = U_{2q}(Y) \cdot D_{40q^2+1}$. By Lemma 4.16 this is $\alpha_0 = D$. So $Y^{2q}(\alpha_{2q-1}) = T_7(D) = 0$. It follows that the annihilator of α_{2q-1} in $Z/2[Y]$ is the ideal (Y^{2q}) . Theorem 4.17 then gives the final assertions. \square

Theorem 4.19 If p is as in Theorem 4.10 then T_p annihilates $DI(q)$. In particular, $X = T_3$ annihilates $DI(q)$.

Proof This is immediate from Theorem 4.10 \square

5 The action of T_3 and T_7 on W_a

This section is the counterpart to section 4 of [1]. In that paper we defined certain subspaces $W1$ and $W5$ of $Z/2[[x]]$; see section 1 of the present paper for a summary. In section 4 of [1] we made $W5$ into a $Z/2[X, Y]$ -module with X and Y acting by T_7 and T_{13} , and we showed the existence of an “adapted basis” $m_{i,j}$ where $D = \sum_{n>0, (n,6)=1} x^{n^2}$ and $m_{0,0} = D^5$. Here we'll derive a similar result with $W5$, T_7 and T_{13} replaced by W_a , T_3 and T_7 .

Since $\chi(3) = \chi(7) = 1$, the commuting operators T_3 and $T_7 : Z/2[[x]] \rightarrow Z/2[[x]]$ stabilize W_a ; see Corollary 2.20. So one can make W_a into a $Z/2[X, Y]$ -module with X and Y acting by T_3 and T_7 . We shall filter W_a by finite-dimensional $Z/2[X, Y]$ -stable subspaces, $W_a(q)$, q running over the powers of 2.

In the paragraph following Definition 2.27 we constructed an isomorphism between W_a and a certain subspace V' of $Z/2[w]$. Using this identification we may speak of the “ w -degree” of an element of W_a . Theorem 2.24 shows that the maps X and $Y : W_a \rightarrow W_a$ lower the w -degree.

Definition 5.1 $W_a(q)$ is the subspace of W_a , of $Z/2$ -dimension $8q^2$, consisting of elements of w -degree $< 40q^2$. (Since X and Y lower the w -degree they stabilize $W_a(q)$.)

Theorem 5.2 The space $DI(q)$ of Definition 4.12, of $Z/2$ -dimension $2q$, is a $Z/2[X, Y]$ -submodule of $W_a(q)$ annihilated by X .

Proof By Theorems 4.17, 4.18 and 4.19, $DI(q)$ has $Z/2$ -dimension $2q$, is annihilated by X and as $Z/2[Y]$ -module is cyclic generated by $\alpha_{2q-1} = Y \cdot D_{40q^2+1}$. So it suffices to show that α_{2q-1} is in $W_a(q)$. But Y lowers the w -degree. \square

We’ll use the results of section 3 to show that the kernel of $X : W_a(q) \rightarrow W_a(q)$ is precisely $DI(q)$. Note that $g(2q) = 4q^2$. So in the language of section 3, $\langle 0, 2q \rangle = w^{40q^2+1}$, and so the w -degree of $(0, 2q)^*$ is $40q^2 + 1$.

Lemma 5.3 Suppose $f \neq 0$ is in $W_a(q)$, with $Xf = 0$. Write f as $(a, b)^* + a$ sum of earlier D_i , as in section 3. Then $a = 0$, and $0 \leq b < 2q$.

Proof If $a > 0$, then by Theorem 3.5, $Xf = (a-1, b)^* + a$ sum of earlier D_i , and so cannot be 0. So $f = (0, b)^* + a$ sum of earlier D_i . By Theorem 3.6, the w -degree of f is the w -degree of $(0, b)^*$. If $b \geq 2q$, then this last w -degree is \geq the w -degree, $40q^2 + 1$, of $(0, 2q)^*$. So f cannot be in $W_a(q)$. \square

Theorem 5.4 The kernel of $X : W_a(q) \rightarrow W_a(q)$ is $DI(q)$.

Proof By Lemma 5.3 this kernel has dimension $\leq 2q$, and we use Theorem 5.2. \square

Corollary 5.5 $DI(1) \subset DI(2) \subset DI(4) \subset \dots$, and the kernel of $X : W_a \rightarrow W_a$ is the union, DI of the $DI(q)$.

Theorem 5.6 The only elements of W_a annihilated by X and Y are 0 and D .

Proof If $(X, Y)f = 0$, f is in DI by Corollary 5.5, and so is in some $DI(q)$.

But $DI(q)$, as $Z/2[Y]$ -module, is isomorphic to $Z/2[Y]/(Y^{2q})$. It follows that the kernel of $Y : DI(q) \rightarrow DI(q)$ has dimension 1 over $Z/2$. \square

Definition 5.7 S_m is the subspace of W_a of dimension $m(m+1)/2$, spanned over $Z/2$ by the $(a, b)^*$ with $a + b < m$.

Note that $S_0 = (0)$ while S_1 is spanned by $(0, 0)^* = D$. So $X \cdot S_1 = Y \cdot S_1 = S_0$.

Lemma 5.8 $X : W_a \rightarrow W_a$ is onto. In fact X maps S_{m+1} onto S_m .

Proof By Theorem 3.5, $X \cdot S_{m+1} \subset S_m$, so it suffices to show that the kernel of $X : S_{m+1} \rightarrow S_m$ has dimension at most $m + 1$. Suppose $f \neq 0$ is in this kernel. The proof of Lemma 5.3 shows that $f = (0, b)^* +$ a sum of earlier D_i , and that the w -degree of f is the w -degree of $(0, b)^*$. But Theorem 3.6 tells us that every element of S_{m+1} has w -degree \leq the w -degree of $(0, m)^*$. So $0 \leq b \leq m$, and the result follows. \square

Theorem 5.9 $Y \cdot S_{m+1} \subset S_m$.

Proof We argue by induction on m , $m = 0$ being clear. Suppose f is in S_{m+1} with $m > 0$. Then Xf is in S_m , so by the induction hypothesis, $X(Yf) = Y(Xf)$ is in S_{m-1} . By Lemma 5.8, there is an h in S_m with $Xh = XYf$. Then $h + Yf$, being in the kernel of X , is $(0, b)^* +$ a sum of earlier D_i for some b . Since $h + Yf$ is in S_{m+1} , $b \leq m$, and it will suffice to show that $b \neq m$. Suppose on the contrary that $h + Yf = (0, m)^* +$ a sum of earlier D_i . Then since f is in S_{m+1} and Y lowers the w -degree, the w -degree of the left hand side is $<$ the w -degree of $(0, m)^*$. But that of the right hand side equals the w -degree of $(0, m)^*$, giving a contradiction. \square

Lemma 5.10 For each m there is an element of DI of the form $(0, m)^* +$ a sum of earlier D_i .

Proof Fix $q > m$. Each $f \neq 0$ in $DI(q)$ can be written as $(0, b)^* +$ a sum of earlier D_i for some b with $0 \leq b < 2q$. Since there are only $2q$ choices for b , and $DI(q)$ has dimension $2q$, the result follows.

Lemma 5.11 $DI \cap S_m$ has dimension m . Furthermore, Y maps $DI \cap S_{m+1}$ onto $DI \cap S_m$.

Proof By Lemma 5.10, $DI \cap S_{m+1} \neq DI \cap S_m$. Now Y maps $DI \cap S_{m+1}$ into $DI \cap S_m$ by Theorem 5.9; Theorem 5.6 shows that the kernel of this map is contained in $\{0, D\}$. So the map is onto, and the dimensions of $DI \cap S_{m+1}$ and $DI \cap S_m$ differ by 1. \square

The machinery is now in place to establish the results analogous to those of section 4 of [1]. The arguments are exactly the same as those that were used

to derive Theorems 4.12, 4.14, 4.15, 4.16, 4.17 and Corollary 4.13 of [1].

Theorem 5.12 *Let f and h be in S_m with $Yf = Xh$. Then there is an e in S_{m+1} with $Xe = f$ and $Ye = h$.*

Corollary 5.13 *There are $m_{a,b}$ in S_{a+b+1} such that:*

- (1) $m_{0,0} = D$.
- (2) $X \cdot m_{a,b} = m_{a-1,b}$ or 0 according as $a > 0$ or $a = 0$.
- (3) $Y \cdot m_{a,b} = m_{a,b-1}$ or 0 according as $b > 0$ or $b = 0$.
- (4) The $m_{a,b}$ are a $Z/2$ -basis of W_a .

Theorem 5.14 *Make W_a into a $Z/2[[X, Y]]$ -module with X and Y acting by T_3 and T_7 . (This is possible since T_3 and T_7 lower the w -degree.) Then the action of $Z/2[[X, Y]]$ on W_a is faithful.*

Theorem 5.15 *If $\chi(p) = 1$, $T_p : W_a \rightarrow W_a$ is multiplication by some u in the ideal (X, Y) of $Z/2[[X, Y]]$. In other words, T_p in its action on W_a is a power series with 0 constant term in T_3 and T_7 .*

We next prove an analogue to Theorem 4.18 of [1]. Since $\chi(11) = -1$, $T_{11}(W_a) \subset W_b$, $T_{11}(W_b) \subset W_a$, and T_{11}^2 stabilizes W_a . We now decompose W_a into a direct sum of 4 summands. The first is spanned by the D_k with $k \equiv 1$ or 9 (40). For the second, $k \equiv 3$ or 27, for the third, $k \equiv 7$ or 23, and for the fourth, $k \equiv 21$ or 29. By Lemma 2.10, the exponents appearing in elements of the first summand are 1 or 9 mod 40, and corresponding results hold for the other summands.

Theorem 5.16 $T_{11}^2 : W_a \rightarrow W_a$ is multiplication by λ^2 for some λ in the ideal (X, Y) of $Z/2[[X, Y]]$.

Proof As in the proofs of Theorem 4.16 and 4.17 of [1] we use the $Z/2[[X, Y]]$ -linearity of our map to show that it is multiplication by some u in (X, Y) and we write u as $a + bX + cY + dXY$ with a, b, c, d in $Z/2[[X^2, Y^2]]$. Let $a = \lambda^2$. We'll show that for each h in W_a , $T_{11}^2(h) = \lambda^2 h$. We may assume that h is in one of the 4 subspaces of the above direct sum decomposition. Suppose for example that it is in the second. Then $T_{11}^2(h)$ and ah are sums of D_k , $k \equiv 3$ or 27 (40), $(bX)h$ is a sum of D_k , $k \equiv 1$ or 9 (40), $(cY)h$ is a sum of D_k , $k \equiv 21$ or 29 (40) and $(dXY)h$ is a sum of D_k , $k \equiv 7$ or 23 (40). Since $T_{11}^2(h) = uh$ is the sum of $\lambda^2 h$, $(bX)h$, $(cY)h$ and $(dXY)h$, and the decomposition is direct, the result follows. \square

Lemma 5.17 *Write the λ of Theorem 5.16 as $cX + dY + \dots$, with c and d in $\{0, 1\}$. Then $c = d = 1$.*

Proof On W_a , $T_{11}^2 = \lambda^2 = cX^2 + dY^2 + \dots$. Applying both sides to D_9 we find that $0 = cD_1 + dD_1$, while applying both sides to D_{41} shows that $D_1 = dD_1$. \square

Theorem 5.18 T_{11}^2 maps S_{m+2} onto S_m for all m . So $T_{11}^2 : W_a \rightarrow W_a$ is onto.

Proof Using the explicit description of the action of T_3 and T_7 on W_a provided by Corollary 5.13 we see that $X^2 + Y^2$ maps S_{m+2} onto S_m . Since $T_{11}^2 = X^2 + Y^2 + \dots$, T_{11}^2 and $X^2 + Y^2$ induce the same map $S_{m+3}/S_{m+2} \rightarrow S_{m+1}/S_m$. So this map is onto for all m , and an induction gives the result. \square

6 $T_{11} : W_b \rightarrow W_a$ is bijective

Since $\chi(11) = -1$, T_{11} maps W_a to W_b and W_b to W_a . In this section we show that $T_{11}(D_k)$ is a sum of D_i , $i < k$, and that $T_{11} : W_b \rightarrow W_a$ is bijective. Let C_k , $(k, 10) = 1$, be $T_{11}(D_k)$.

Lemma 6.1 *If $k < 120$ then:*

- (a) *When $k \equiv 1, 3, 7$ or $9 \pmod{20}$, C_k is a sum of D_i , $i < k$.*
- (b) *When $k \equiv 11 \pmod{20}$ (resp. $19, 13, 17 \pmod{20}$), $C_k = D_j +$ a sum of D_i , $i < j$, where $j = k - 10$ (resp. $k - 10, k - 6, k - 14$).*

Proof Sketch Using Corollary 2.20 and Lemmas 2.22 and 2.23 we see that it suffices to prove (a) when $k \equiv 9 \pmod{20}$, i.e. when k is in $\{9, 29, 49, 69, 89, 109\}$. In fact the corresponding C_k are $0, 0, D_{19} + D_{11}, D_{39} + D_{31}, D_{11}$ and D_{39} . To calculate C_{109} , for example, we use Corollary 2.20 and Lemma 2.23 to see that it is a sum of D_i with each i in $\{31, 39, 71, 79, 111\}$. Examining the expansions of D_{109} and the D_i and arguing as in the proof of Theorem 2.26 we find that $C_{109} = D_{39}$. (b) follows from the more precise results:

- (b1) $C_{11}, C_{31}, C_{51}, C_{71}, C_{91}$ and C_{111} are $D_1, D_{21}, D_{41} + D_9, D_{61} + D_{29}, D_{81} + D_{41} + D_9, D_{101} + D_{61} + D_{21}$.
- (b2) $C_{19}, C_{39}, C_{59}, C_{79}, C_{99}$ and C_{119} are $D_9, D_{29}, D_{49} + D_9, D_{69} + D_{29}, D_{89} + D_{49} + D_9, D_{109} + D_{69} + D_{29} + D_{21}$.
- (b3) $C_{13}, C_{33}, C_{53}, C_{73}, C_{93}$ and C_{113} are $D_7, D_{27} + D_3, D_{47}, D_{67} + D_{43} + D_{27}, D_{87} + D_{47}, D_{107} + D_{83} + D_{67} + D_{43} + D_{27}$.
- (b4) $C_{17}, C_{37}, C_{57}, C_{77}, C_{97}$ and C_{117} are $D_3, D_{23} + D_7, D_{43}, D_{63} + D_{47} + D_{23} + D_7, D_{83} + D_{43}, D_{103} + D_{87} + D_{63} + D_{47} + D_{23}$.

To calculate C_{93} , for example, we use Corollary 2.20 and Lemma 2.22 to see that it is a sum of D_i where each i is in $\{7, 23, 47, 63, 87\}$. Examining the expansions of D_{93} and the D_i and arguing as in the proof of Theorem 2.26 we find that $C_{93} = D_{87} + D_{47}$. \square

Lemma 6.2 *For u in $Z/2[[x]]$, $T_{11}(uG^{24})$ is the sum of $G^{2i}T_{11}(uG^{2j})$ where (i, j) runs over the 9 pairs $(12, 0), (8, 4), (4, 8), (6, 2), (2, 6), (9, 1), (1, 9), (3, 3), (1, 1)$.*

Proof We argue as in the proof of Lemma 2.25. Let U be the 2 variable polynomial $(A + B)^{12} + A^6B^2 + A^2B^6 + A^9B + AB^9 + A^3B^3 + AB$. Then $U(F(x^{11}), F(x)) = 0$; this is the level 11 modular equation for F . Replacing x by x^5 we find that $U(G(x^{11}), G(x)) = 0$. Now let L be an algebraic closure of $Z/2$. We have 12 imbeddings $\varphi_k : Z/2[[x]] \rightarrow L[[x^{1/11}]]$, the first of which takes f to $f(x^{11})$, while each of the others takes f to $f(\lambda x^{1/11})$ for some λ in L with $\lambda^{11} = 1$. Replacing x by $\lambda x^{1/11}$ in the identity $U(G(x^{11}), G(x)) = 0$ and using the symmetry of U we find that each $U(\varphi_k(G), G)$ is 0. Squaring and expanding we find that $\varphi_k(G^{24})$ is the sum of the $G^{2i}\varphi_k(G^{2j})$ where (i, j) runs over the 9 pairs above. Now the definition of T_{11} shows that if f is in $Z/2[[x]]$, $T_{11}(f)$ is the sum of the $\varphi_k(f)$. Multiplying the k^{th} of our identities by $\varphi_k(u)$ and summing we get the result. \square

Lemma 6.3 *The conclusions (a) and (b) of Lemma 6.1 hold for all k with $(k, 10) = 1$.*

Proof We argue by induction on k . For $k < 120$, Lemma 6.1 applies. Suppose $k = n + 120$ with $n > 0$. By Lemma 6.2, C_{n+120} is a sum of 9 terms each corresponding to one of the 9 pairs (i, j) of the lemma. These terms are $G^{24}C_n, G^{16}C_{n+40}, G^8C_{n+80}, G^{12}C_{n+20}, G^4C_{n+60}, G^{18}C_{n+10}, G^2C_{n+90}, G^6C_{n+30}$ and G^2C_{n+10} . The induction hypothesis shows that the term corresponding to (i, j) is a sum of D_m with $m < (n + 10j) + 10i$. Since $i + j \leq 12$, each $m < n + 120$, and in particular (a) holds for k . We turn to (b). For each of the last 6 terms in our sum, $i + j \leq 10$. So the corresponding term is a sum of D_m with $m < n + 100$. Now consider the first 3 terms. Suppose for example that $n \equiv 17 \pmod{20}$. By the induction hypothesis, each of $G^{24}C_n, G^{16}C_{n+40}$ and G^8C_{n+80} is D_{n+106} + a sum of D_m with $m < n + 106$. It follows that C_{n+120} itself is D_{n+106} + a sum of D_m with $m < n + 106$. The proof of (b) when $n \equiv 11, 19$ or $13 \pmod{20}$ is identical. \square

Theorem 6.4 *If $(k, 10) = 1$, $T_{11}(D_k)$ is a sum of D_i with $i < k$. Furthermore, $T_{11} : W_b \rightarrow W_a$ is bijective.*

Proof Lemma 6.3 gives the first result. Also the D_k where $k > 0$ and $\equiv 11, 19, 13$ or $17 \pmod{20}$ form a basis of W_b , while the corresponding $D_{k-10}, D_{k-10}, D_{k-6}$ and D_{k-14} form a basis of W_a . So Lemma 6.3 gives the second result as well. \square

Corollary 6.5 *$T_{11} : W_a \rightarrow W_b$ is onto.*

Proof By Theorem 5.18, T_{11} maps the subspace $T_{11}(W_a)$ of W_b onto W_a , and we use the last theorem to see that $T_{11}(W_a)$ is all of W_b . \square

Corollary 6.6 *Let λ be as in Theorem 5.16. Then in its action on $W = W_a \oplus W_b$, T_{11}^2 is multiplication by λ^2 . (This follows from Corollary 6.5 and Theorem 5.16.)*

7 The algebra \mathcal{O} acting on W_a

Take λ in (X, Y) as in Theorem 5.16, and let $U : W \rightarrow W$ be the map $h \rightarrow \lambda(X, Y)h + T_{11}(h)$. Since U is $Z/2[[X, Y]]$ -linear, and U^2 annihilates W (by Corollary 6.6), W has the structure of $Z/2[[X, Y]][\varepsilon]$ module with $\varepsilon^2 = 0$, ε acting by U , and X and Y by T_3 and T_7 . Let \mathcal{O} be the local ring $Z/2[[X, Y]][\varepsilon]$.

Lemma 7.1 *The only element of W_b annihilated by ε is 0.*

Proof Suppose $\varepsilon h_b = 0$, with h_b in W_b . Then $T_{11}(h_b) = \lambda(T_3, T_7)h_b$. Since the first of these is in W_a and the second in W_b , $T_{11}(h_b) = 0$. By Theorem 6.4, $h_b = 0$. \square

In Corollary 5.13 we constructed a $Z/2$ -basis $m_{i,j}$ of W_a “adapted to $X = T_3$ and $Y = T_7$.” Since $T_{11} : W_b \rightarrow W_a$ is bijective, this basis pulls back under T_{11} to a basis $n_{i,j}$ of W_b “adapted to T_3 and T_7 .”

Theorem 7.2

- (1) *The $n_{i,j}$ and the $\varepsilon n_{i,j}$ form a $Z/2$ -basis of W .*
- (2) *\mathcal{O} acts faithfully on W .*
- (3) *Each T_p , $p \neq 2$ or 5, acts on W by multiplication by some $r + t\varepsilon$ with r and t in $Z/2[[X, Y]]$.*

Proof The arguments are just like those giving Theorems 5.2, 5.3 and 5.4 of [1], using the basis $n_{i,j}$ of W_b . \square

Theorem 7.3

- (1) *If $\chi(p) = 1$, $T_p : W \rightarrow W$ is multiplication by some t in the maximal ideal (X, Y) of $Z/2[[X, Y]]$.*
- (2) *If $\chi(p) = -1$, $T_p : W \rightarrow W$ is the composition of T_{11} with multiplication by some t in the maximal ideal (X, Y) of $Z/2[[X, Y]]$.*

Proof Suppose $\chi(p) = 1$. By Theorem 5.15, $T_p : W_a \rightarrow W_a$ is multiplication by some t in the maximal ideal (X, Y) . Since $T_{11} : W_a \rightarrow W_b$ is onto, $T_p : W_b \rightarrow W_b$ is multiplication by the same t , and we get (1). Now T_{11} is multiplication by $\lambda + \varepsilon$. Suppose $\chi(p) = -1$. By Theorem 7.2, T_p is multiplication by some $r + t\varepsilon$ with r and t in $Z/2[[X, Y]]$. Then $T_p + tT_{11}$ is multiplication by $r + \lambda t$. Since

$T_p + tT_{11}$ and multiplication by $r + \lambda t$ map W_b into W_a and W_b respectively, $T_p + tT_{11} = 0$, giving (2). \square

Corollary 7.4 *If $p \neq 2$ or 5 and $(k, 10) = 1$, then $T_p(D_k)$ is a sum of D_i with $i < k$.*

Proof We have seen that this holds when $p = 3, 7$ or 11; Theorem 7.3 then gives the general result. \square

Theorems 7.2 (2) and 7.3 tell us that when we complete the Hecke algebra generated by the T_p acting on W with respect to the maximal ideal generated by the T_p , then the completed Hecke algebra we get is just the (non-reduced) local ring \mathcal{O} . This is completely analogous to the results of [1]; see Theorems 5.3 and 5.5 of that paper.

References

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